# CG Image Generation of Four-Dimensional Origami

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Abstract

To produce four-dimensional (4-D) origami, we fold a solid material along flat planes in a 4-D space. A 4-D space has a fourth axis that is perpendicular to a three-dimensional space. Because a computer graphics (CG) image of a 4-D origami must be drawn on a three-dimensional screen to visualize it, we will produce its CG image using a 4-D painter's algorithm with a stereogram. First, we will show how to fold a solid material in a 4-D space. After defining front and a back sides of this solid in a 4-D space, we will make mountain and valley folds and thereby produce a "4-D Noshi" CG image. Secondly, we will show how to fold a regular tetrahedron flat by bisectors of its dihedral angles and make a 4-D bird base from a double tetrahedron. Finally, we will produce CG images of this 4-D bird with opened wings using a stereogram.

Keywords: Four-dimensional origami, Four-dimensional painter's algorithm, Four-dimensional space, Stereogram, View space

# ４次元折り紙の CG 画像生成

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**概要**

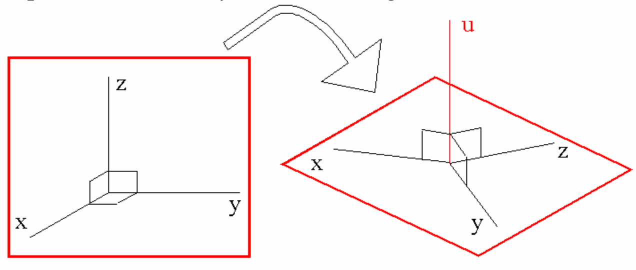
4次元折り紙では，4次元空間において立体を折り面で折る．4次元空間とは，この3次元空間とそれに直交する第4軸により張られた空間である．4次元折り紙の CG 画像は3次元スクリーンで描くのが合理的なので，本論では4次元 CG 画像を4次元ペインタアルゴリズムとステレオグラムを用いて生成する．最初に，4次元空間における立体の折りたたみについて説明する．この立体の表裏を定義すると，折り紙の山折りと谷折りが決まる．それら用いて「4次元のし」を折り，ステレオ画像を生成する．次に，正4面体を稜角2等分折りにより折りたたみ，それを利用して二重4面体から4次元鶴を折る．最後に，この4次元鶴が翼を開いたところの CG ステレオ画像を数種類生成し，本手法の有効性を示す．

キーワード:　4次元空間，　4次元折り紙，　4次元ペインタアルゴリズム，　3次元スクリーン，　ステレオグラム

## 1. Introduction

Origami has much to offer as an instrument for experimenting with scientific and educational ideas. Several studies of computer processing of origami have been reported. Uchida and Itoh [1] explored a method to derive a sequence of origami folding processes. Its data structure is basically represented as a binary tree. Miyazaki et al. [2] developed a virtual manipulation system for folding origami. A three-dimensional (3-D) origami is made of two-dimensional sheets of paper that are folded in 3-D space. Similarly, a four-dimensional (4-D) origami is made of polyhedra, such as tetrahedra and cubes. Those are folded in a 4-D space in which a fourth axis, the *u*-axis, is perpendicular to a usual 3-D *xyz*-space. Figure 1 shows a model of this 4-D space in which the basic plane represents the 3-D space. We call this 3-D space the u=0 hyper-plane. Kawasaki [3] reported about high-dimensional flat origamis. Miyazaki [4] showed how to fold a 4-D analogue of "Noshi" from a regular octahedron, which is a dual of a cube. Those studies do not address two paper surfaces, for the sake of simplicity.

First, for CG image generation of 4-D objects, we propose 4-D painter's algorithm. Using that algorithm, foreground objects are painted on top of background objects on a view space. We construct the view space using a stereogram. Secondly, we show how to fold a solid along a flat plane. By cutting and folding a parallelepiped, we can produce 4-D origamic architecture [5]. Its CG image is represented by two kinds of hidden lines. Thirdly, we define the front and a back of an origami material in a 4-D space. Then we can make both mountain and valley folds in 4-D space and show a CG image of a 4-D origami "Noshi." Finally, we show how to fold a 4-D origami bird from a double tetrahedron. Using a stereogram, we make CG images of a 4-D bird when it is opening its wings. Texture mapping on the surfaces of these opening wings gives on impression of solidity of the 4-D origami bird.




(a) 3-D space (b) 4-D space

Figure 1. 3-D space and 4-D space

## 2. Basic algorithms for four-dimensional CG

### 2.1 Four-dimensional painter's algorithm

We propose a 4-D painter's algorithm. This algorithm is a method of projecting the 4-D objects on a view space that is a 4-D analogue of a view plane. Let us put a small hyper-sphere near the view space and a large hyper-sphere far away from the view space, as shown in Fig. 2(a), and project them on the view space. We begin with the large hyper-sphere given as

$$x^2 + y^2 + z^2 + (u\text{-}3)^2 = 2,$$

whose cross section is a sphere. With decreasing *u* from 3 to $3\text{-}\sqrt{2}$ , we paint these spheres given as

$$x^2 + y^2 + z^2 = 2\text{-}(u\text{-}3)^2$$

successively to the view space.

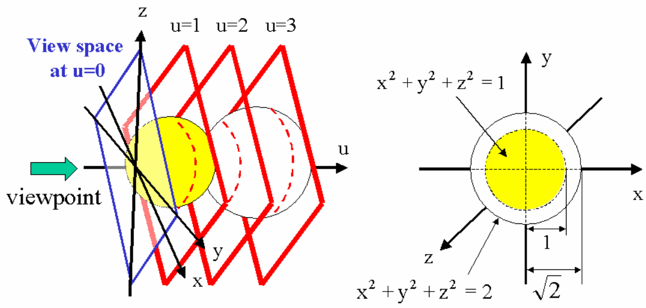
Next, we project the small hyper-sphere given as

$$x^2 + y^2 + z^2 + u^2 = 1 .$$

Decreasing *u* from 1 to 0, we paint its cross sections, given as

$$x^2 + y^2 + z^2 = 1\text{-}u^2,$$

successively on top of the paint background. The resultant sphere we have painted looks like an egg, as shown in Fig. 2(b). Because we cannot view its interior, we paint this sphere translucent.




(a) Two hyper-spheres and their cross sections (b) View space

Figure 2: Parallel projection

### 2.2 Stereogram for view space

Figure 2(b) shows that we project 4-D objects onto the view space. It is easy for us to imagine its 3-D structure, but hard to imagine its 4-D structure when 3-D objects are drawn in a view plane. For 4-D CG image generation, we realize a view space using a holographic stereogram [6]. In this paper, we use a simpler stereogram method to represent the view space.

Stereogram is a picture that gives the impression of solidity. Various methods for stereograms exist: a parallel method, a crossing method, use of anaglyphs, and stereoscopy. We can make a digital camera set for anaglyphs with which a pair of photographs – one for each eye – are taken. Figure 3 shows an anaglyph image of origamic architecture [5]. Using eyeglasses with red and blue filters for anaglyphs, we can view this picture as if it were a 3-D image.

Parallel and crossing methods are simpler for CG image generation. Figure 4 shows a pair of crossing method images of a parallelepiped, which is rotated along the *xz* plane by a right angle. We shall give a brief explanation on the transformation of objects in 4-D space in the next section.

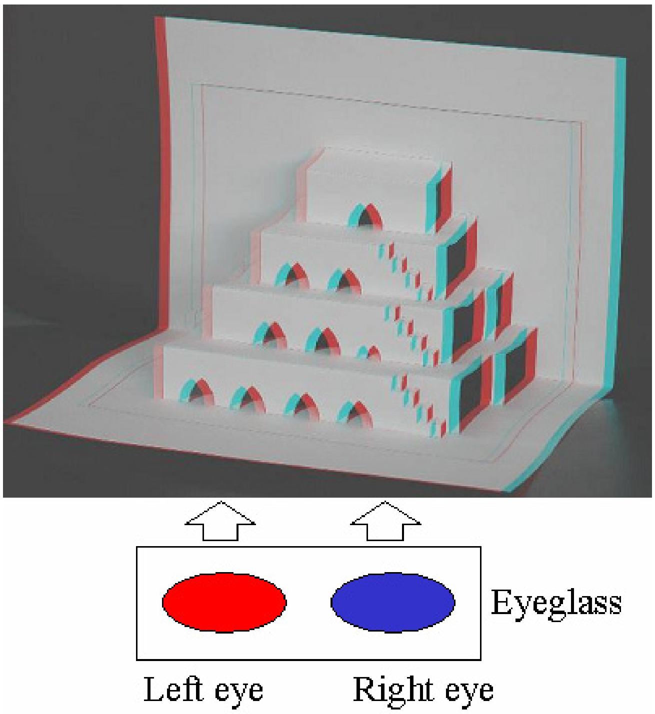



Figure 3: Anaglyphic image of origamic architecture [5]

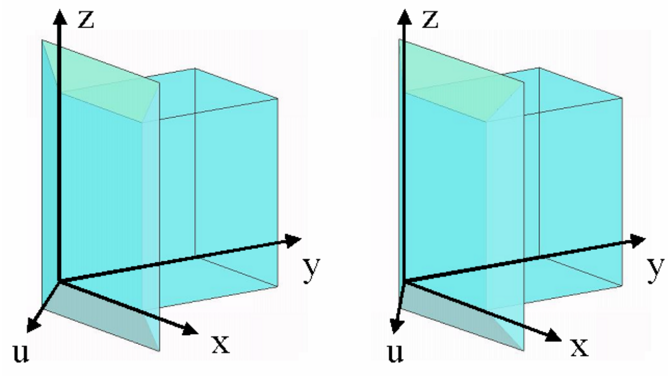



Figure 4: Folding of parallelepiped

### 2.3 Rotation in a two-dimensional plane

A 3-D origami is made of a sheet of paper that is folded in 3-D space. Presume that a rectangular paper is put on the z=0 plane and folded about a fold line. In Fig. 5(a) a point **p**=(x, y, z) for y>0 is rotated about a fold line $\ell$ by an angle $\theta$. Its new position **p'**=(x', y', z') is given as

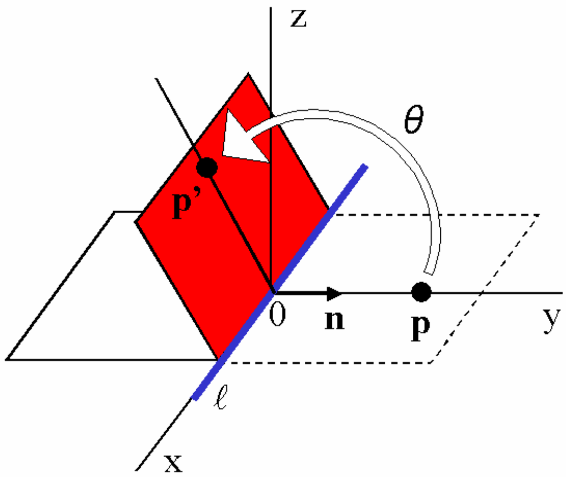
$$\mathbf{p}' = \mathbf{p}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \quad (1)$$

The transformation matrix is a rotation in the *yz* plane.

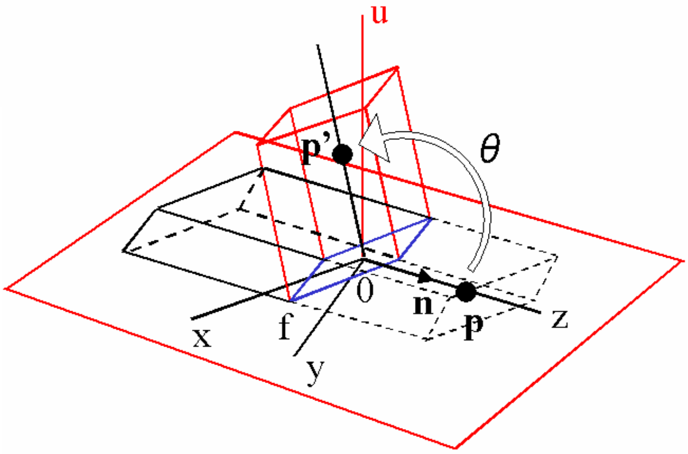
The 4-D origami comprises polyhedra such as tetrahedra and cubes. Figure 4 shows the presumption that a parallelepiped is put on the u=0 hyper-plane and folded about a fold plane. In Fig. 5(b) a point **p**=(x, y, z, u) for z>0 is rotated about a fold plane f by an angle $\theta$. Because a normal vector **n** to f is parallel to the z-axis, the rotation is in the *zu* plane. Its new position **p'**=(x', y', z', u') is determined by multiplication by a matrix [7] as

$$\mathbf{p}' = \mathbf{p}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \quad (2)$$

We apply this folding in 4-D to produce 4-D origamic architecture [5]. We must represent two kinds of hidden lines of its image to make its CG wire-frame image.




(a) 3-D origami




(b) 4-D origami

Figure 5: Folding of 3-D and 4-D origami

### 2.4 Hidden lines and origamic architecture

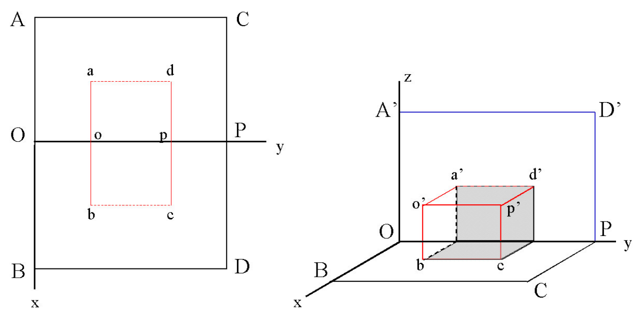
We have shown how to project 4-D objects onto the view space using the 4-D painter's algorithm. We cannot view the contents of 3-D objects from outside. In wire-frame drawings, all parts of objects are always displayed but projected 4-D objects can easily become an indistinguishable mess of line segments. Let us consider the problem of removing those lines that are hidden by parts of the object.

Figure 6(a) shows a cutting and folding pattern of the simplest 3-D origamic architecture. In its completed figure of Fig. 6(b), hidden lines represented by broken lines exist. Figure 7(a) shows a pattern of simplest 4-D origamic architecture, in which four sides of the red parallelepiped are cut and its three faces of abcd, opqr, and efgh are creases. After folding, we obtain the completed figure of Fig. 7(b), which depicts two kinds of hidden lines. Broken lines are apparent because the figure is visible if rotated properly, but dot-dashed lines are not visible because they are inside of the 3-D figure [7].

## 3. Four-dimensional origami

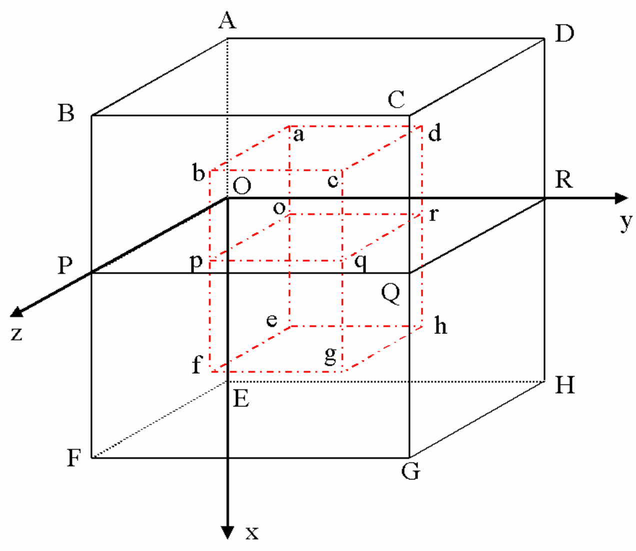
### 3.1 Mountain and valley folds

We take a solid in a u=0 hyper-plane and move this solid a bit upward in the direction of the *u* axis. In this pair of solids, we regard the upper solid as a front side and the lower one as a back side; we define its normal vector **n** as a direction of the relative movement (0,0,0,1). Let us call such a pair of solids a 4-D origami. Then folds come as two types: mountain folds, which are convex, and valley folds, which are concave. Let us put a regular octahedron in the u=0 hyper-plane with the front painted white and the back painted red.

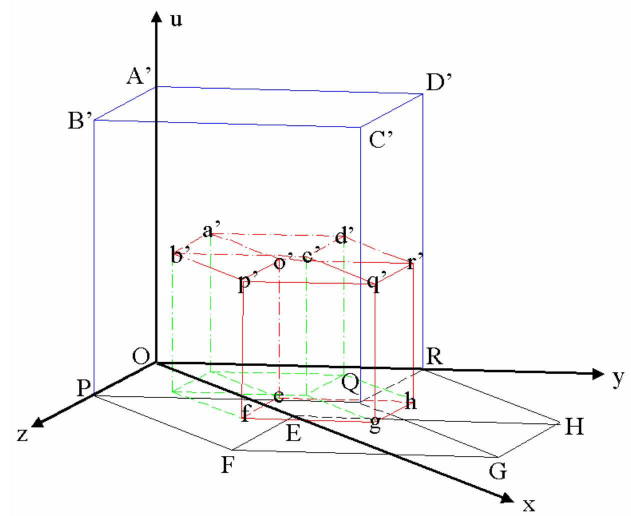



(a) Pattern (b) Completed figure

Figure 6: 3-D origamic architecture and hidden lines




(a) Pattern




(b) Completed figure

Figure 7: 4-D origamic architecture and hidden lines

Figure 8(a) shows the 4-D image where this octahedron is mountain-folded by an angle $\theta$ less than a right angle. An image of the folded octahedron is shown in Fig. 8(b) where the point C' moves to the origin from the point C when the viewpoint is set above in the u axis shown in Fig. 8(a). When rotated by an angle $\theta$ larger than a right angle, as show in Fig. 8(c), the rotated part of the octahedron turns red, as shown in Fig. 8(d).

We fold an origami "Noshi" from a square after alternately making mountain folds and valley folds. Similarly, we can make a 4-D origami "Noshi" from a 4-D origami of a regular octahedron [4]. Figure 9 shows an image of 4-D origami "Noshi" by the crossing method of stereograms. To view the entire 4-D "Noshi", we render it translucent and paint its front parts white and its back part red.

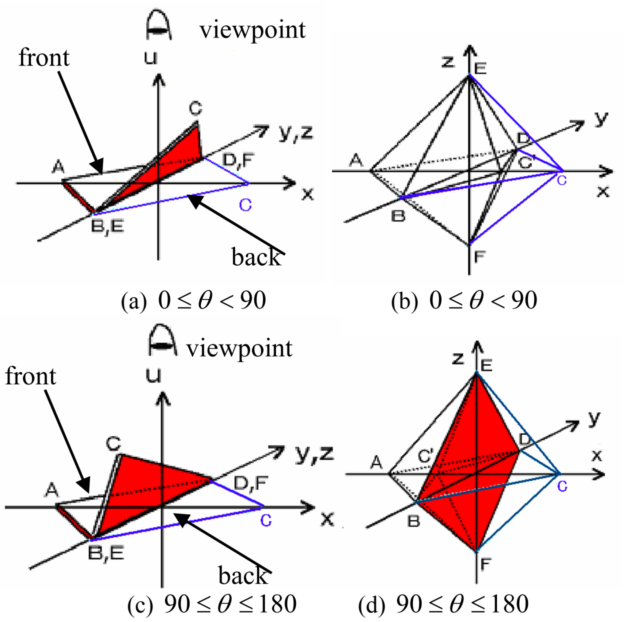



Figure 8: Folded 4-D origami of a regular octahedron and its projection

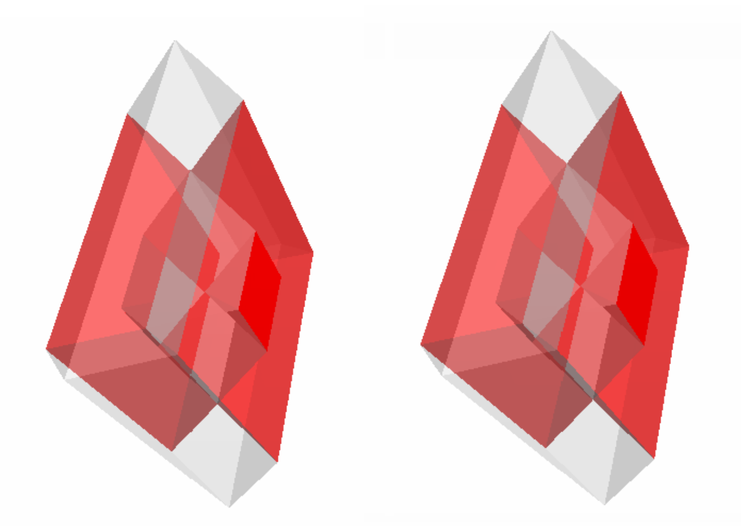


Figure 9: 4-D "Noshi" by crossing method

### 3.2 Folding tetrahedron

We will show how to fold a regular tetrahedron using the incenter theorem of a tetrahedron. Figure 10(a) shows this tetrahedron ABCD, where I is the incenter, M and N are the midpoints of AC and CD, respectively, and O, S and T are the tangency points at which the inscribed sphere touches the faces of ABCD. Each time the tetrahedron is folded, it is divided into two or more small tetrahedra. Those new small tetrahedra lie at different heights along the $u$ direction and are represented by a tree structure with nodes [1]. In Table 1, heights of the small tetrahedra are given as integral values related to a scale $\varepsilon$ of the thickness of the 4-D origami material, whose information indicates the stacking order of the small tetrahedra along the u direction. The procedure for folding the tetrahedron ABCD is as follows [8]:

(1) Fold the top half of ABDM to the bottom BCDM (Fig. 10(a)).

(2) Inside reverse-fold BCMI and DCMI while crimp-folding CTMI (Fig. 10(b)).

(3) Complete the fold (Fig. 10(c)).

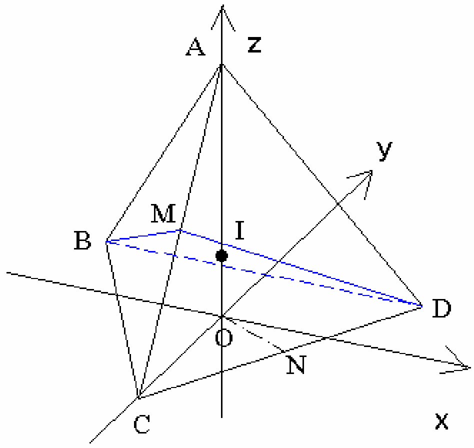
First, the vertex A is superposed to C in the 4-D space. Then the midpoint M is superposed to N and the tangency point T to the origin O. We can see that the four flat faces of the regular tetrahedron lie on the basal plane BCD and that the other two tangency points are superposed to O.

We continuously fold the regular tetrahedron ABCD by simultaneously rotating small tetrahedra from 4 to 11 in Table 1 in the 4-D space. Let us rotate MABD around the fold plane IBD by an angle θ and IBCM by an angle $\phi$ simultaneously and move the point M to M' (N). Using the condition AM'= CM'=AM=CM, we obtain $\tan(\phi/2) = 2\tan(\theta/2)$ and a diagram of $\phi$ as a function of θ, as shown in Fig. 11. That relationship suggests that the regular tetrahedron is continuously folded.

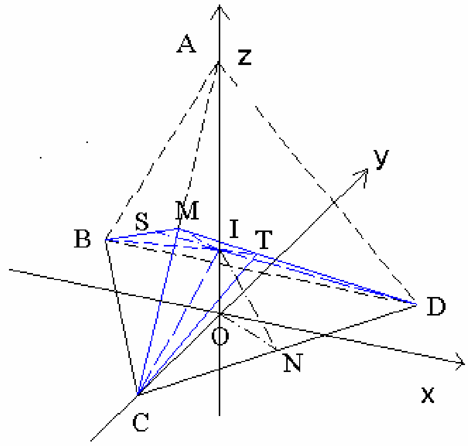
Table 1: Data structure of the tetrahedron

| | Tetrahedron | Height | Father | Sons | Front / Back |
|---|---|---|---|---|---|
| 1 | ABCD | 0 | Root | 2,3 | Front |
| 2 | MBCD | 0 | 1 | 4,5,6,7 | Front |
| 3 | MABD | 7ε | 1 | 8,9,10,11 | Back |
| 4 | IBCD | 0 | 2 | | Front |
| 5 | ICDO | 1ε | 2 | | Back |
| 6 | ICNO | 2ε | 2 | | Front |
| 7 | IBCN | 3ε | 2 | | Back |
| 8 | IA'BM | 4ε | 3 | | Front |
| 9 | IA'MO | 5ε | 3 | | Back |
| 10 | IA'DO | 6ε | 3 | | Front |
| 11 | IA'BD | 7ε | 3 | | Back |

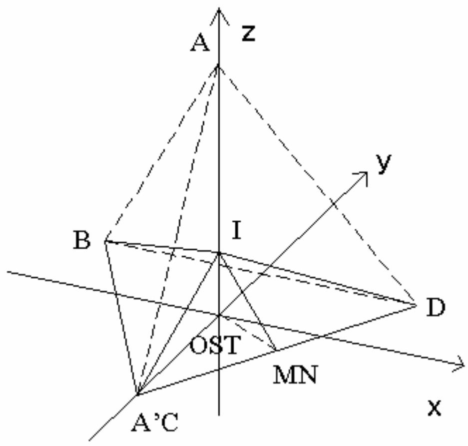
A → A'(C), M → N, S, T → O




(a)




(b)




(c)

Figure 10: Folding of regular tetrahedron

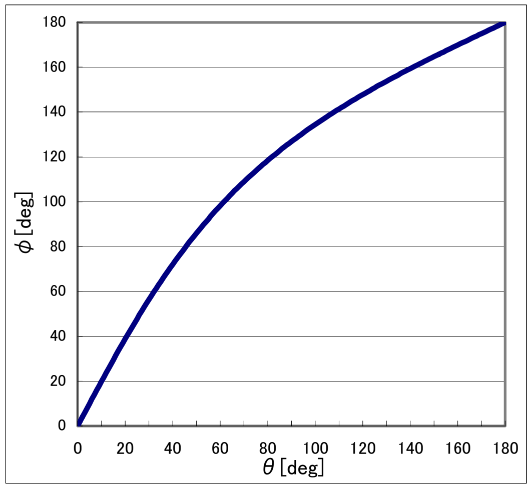



Figure 11: Relation to the fold-angle of the regular tetrahedron

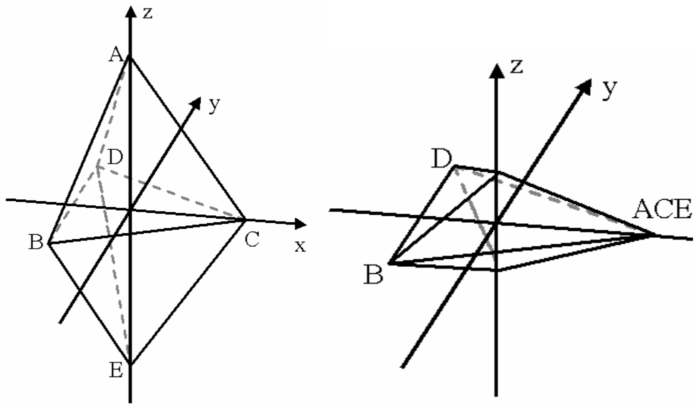
### 3.3 Four-dimensional bird

Traditional origami is usually folded from a square paper. Classic examples are the “yakko” (serving man) and the “orizuru” (crane). An orizuru has a neck and a tail with mobile wings. Those are made from four corners of a square. It is a challenging problem to fold a 4-D crane. For simplicity, let us fold a 4-D bird from a double tetrahedron, as shown in Fig. 12(a).

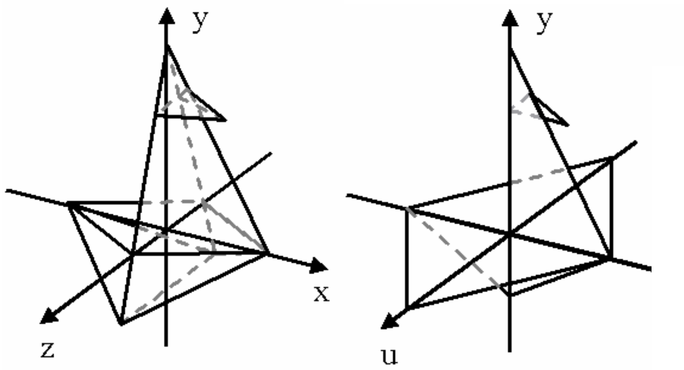
A tetrahedron is folded using the incenter theorem so as to contact three faces (z>0) to the basic plane (z=0) [8]. After folding both the upper and the lower tetrahedron in the same way, we obtain a base of the bird shown in Fig. 12(b). From this base, we can fold a bird with wings. They disappear, as shown in an image of Fig. 12(c), because it is drawn in the u=0 hyper-plane when those wings open along the *u*-axis. We can see wings as shown in Fig. 12(d), whereas we can hardly see the body because it has no width when it is drawn in the z=0 hyper-lane. We can resolve this problem using the stereogram.

Four-dimensional origami can be portrayed in the view space using a stereogram. Then we project the body of a 4-D origami bird onto the u=0 hyper-plane, as shown in Fig. 1(b), and present its opening wings using a stereogram. Figure 13 is an image of a wire frame model of the 4-D origami bird; Fig. 14 is an image of its solid model with translucent wings.

Figure 15 shows anaglyph images of the 4-D origami bird with opened wings. As we can readily imagine, the wire frame model is suitable for representing the structure of the 4-D origami bird. On the other hand, the anaglyph method seems not to work good because numerous flat faces exist. Figure 16 shows that this problem is resolved when wings are texture-mapped.




(a) Double tetrahedron (b) Bird base




(c) Projection on u=0 (d) Projection on z=0

Figure 12: Four-dimensional bird

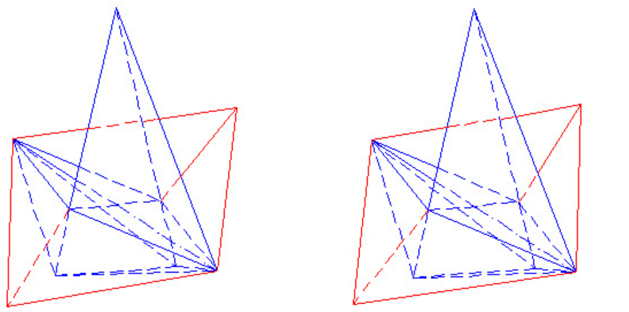


Figure 13: 4-D bird depicted as a wire frame model

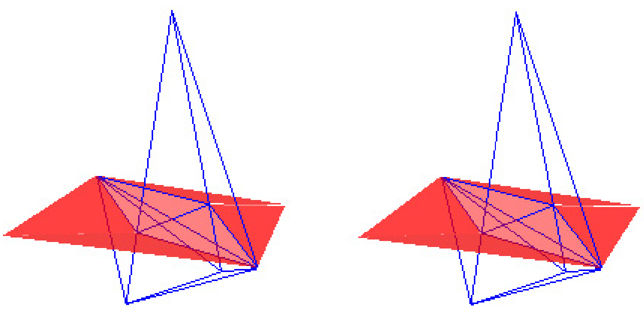
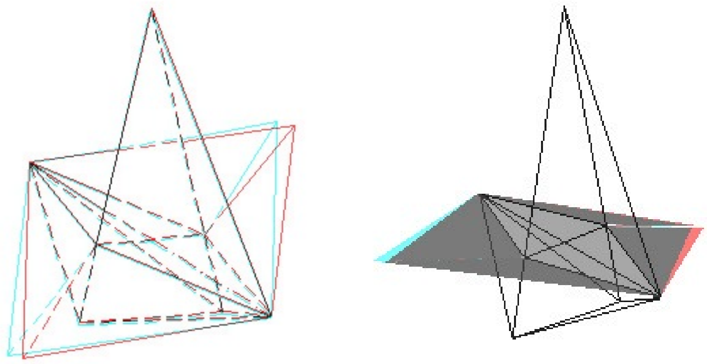
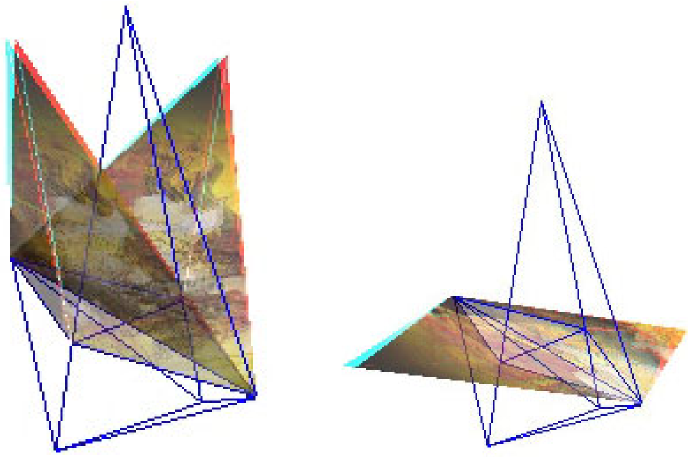


Figure 14: 4-D bird by solid model



(a) Wire frame model (b) Solid model

Figure 15: Anaglyph image of a 4-D bird



(a) (b)

Figure 16: Anaglyph image of 4-D bird using texture mapping

## 4. Conclusions

We have studied 4-D CG algorithms and 4-D origamis and applied them to produce CG image generation of 4-D origami [9]. We have obtained the following results:

(1) We proposed the 4-D painter's algorithm with the view space presented using a stereogram. Subsequently, we applied this algorithm to produce a CG image of 4-D origamic architecture.

(2) We defined a front and a back of 4-D origami paper for mountain and valley folds and applied this method to produce a CG image of 4-D origami "Noshi".

(3) We partly proved that a regular tetrahedron is continuously folded using the incenter theorem of a tetrahedron and made a 4-D bird base from a double tetrahedron. Using the above bird base, we folded a 4-D origami bird and made a 4-D origami flapping bird.

(4) We showed that a stereogram with texture

mapping is helpful to imagine a complex structure of 4-D origami.

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