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3D Realization of Penrose Polygons Using Non-Rectangularity Trick

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Abstract

This paper generalizes an impossible figure called a "Penrose triangle" to regular polygonal figures and proposes a method for constructing real 3D objects from such figures. Two well-known tricks, namely the curved surface trick and the discontinuity trick, have been used to construct 3D objects from impossible figures. However, these tricks are very sensitive to viewpoint. The proposed method uses only planar faces and connects them wherever they appear to be connected. The resulting objects are thus less sensitive to viewpoint in the sense that the impression of impossibility does not disappear when the viewpoint slightly changes. Moreover, the constructed objects have the same symmetry as that of the original figures, preserving their elegance.

1 Introduction

A class of images called "impossible figures" can be imagined as 3D structures but seem impossible to physically exist [1, 2]. The Swedish graphic artist Oscar Reutersvärd, who drew many beautiful impossible figures around the middle of the twentith century, is called the father of impossible figures, even though their inventor is unknown [3]. Various impossible figures have been created, attracting the attention of both scientists and artists.

Visual scientists study impossible figures to understand how the human brain perceives 3D objects. Gregory [4], for example, constructed a 3D model of an impossible figure and discussed the differences between seeing 3D objects and their 2D images. Huffman [5] and Clowes [6] characterized the local features of figures to explain why the brain can imagine the 3D structures despite their impossibility.

Artists have used impossible figures in their artwork. For example, the Dutch artist M. C. Escher [7, 8] drew miraculous buildings using impossible figures. The Japanese artist Fukuda [9] constructed real 3D structures from Escher's images and thus developed 3D trick art.

One of the simplest impossible figures is the "Penrose triangle" [10], in which three rectangular rods are connected at their ends, forming a triangular ring. The rods are connected in such a manner that the structure seems unrealizable as a real 3D object. The Penrose triangle is one of the most beautiful impossible figures because of its simplicity and symmetry.

Although such figures are called "impossible", their 3D realization may be possible. Two tricks, namely the curved surface trick and the discontinuity trick, are widely used to construct 3D objects from impossible figures. The Penrose triangle has been realized as a 3D object using these tricks [4, 2, 11]. However, the resulting objects are very sensitive to viewpoint; even a slight change in viewpoint ruins the trick and the sense of impossibility disappears.

3D objects can be constructed from some impossible figures without using these two tricks [12, 13]. In this method, a system of linear equations and inequalities is solved; if the system has solutions, a 3D object associated with each of the solutions can be constructed. However, a general application of this method results in an object without symmetry even if the original figure is highly symmetric.

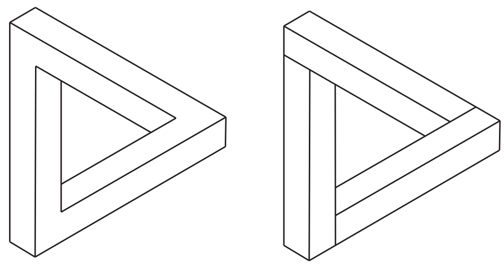
In this paper, we generalize the Penrose triangle to regular polygons and propose a method for constructing the associated 3D objects without using the curved surface trick or the discontinuity trick. The resulting 3D objects are artistically elegant in the sense that they inherit the rotational symmetry of the original impossible figures.

The rest of this paper is organized as follows. Existing methods for realizing 3D objects from the Penrose triangle are reviewed in Section 2. A rectangular version of the Penrose triangle by Draper and its 3D realization are presented in Section 3. The Penrose triangle is generalized to general regular polygons and a method for 3D realization that inherits the original symmetry is proposed in Section 4. The relation between our method and a traditional graphical method [14, 15, 16] is discussed in Section 5, and concluding remarks are given in Section 6.

2 Penrose Triangle and Two 3D Realization Methods

Figure 1 shows two versions of the Penrose triangle. Panel (a) shows a ring structure composed of three rectangular rods connected at their terminals. Panel (b) shows the same structure except that the joint lines are also drawn along the lines at which two rods are connected. These figures give the impression that the rods are connected at right angles and thus seem to be physically impossible because the accumulation of three right angles cannot form a closed loop. Even if the rods are connected by 60-degree inner angles, they are twisted and thus the structure appears to be physically impossible. These figures were popularized by Lionel and Roger Penrose [10], after whom they are named. Reutersvärd drew essentially the same structure composed of many cubes [3].

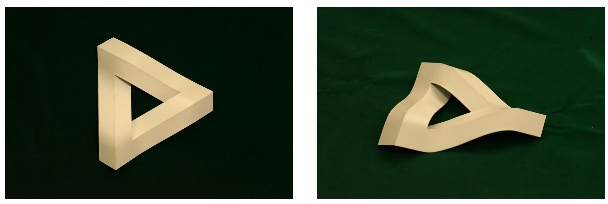
The Penrose triangle in Figure 1(a) can be constructed as a real 3D object using curved surfaces, as shown in Figure 2, where (a) shows the object



(a) (b)

Figure 1: Penrose triangle drawn (a) without and (b) with joint lines.

seen from a special viewpoint and (b) shows the same object seen from a general angle. This trick is called the curved surface trick. The faces of the object are curved but the boundary appears straight when seen from the special viewpoint, giving the impression of a realized Penrose triangle. This trick was used by computer scientist Elber [11] and sculptor Hamaekers [2].



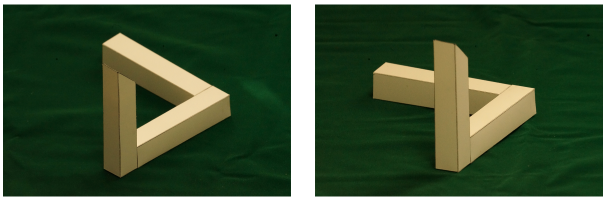
(a) (b)

Figure 2: Realization using curved surface trick.

Note that this trick can also be used to realize the image in Figure 1(b). To do this, we cut the object in Figure 2 into three parts and reconnect them. However, this construction is less interesting because any 3D object without joint lines can be trivially converted to one with joint lines by cutting along the joint lines and gluing along them.

The other version of the Penrose triangle, shown in Figure 1(b), can be realized as a 3D object as shown in Figure 3, where (a) is a photograph taken from a special viewpoint and (b) shows the same object from a general viewpoint. As shown in Figure 3(b), the object is not closed; there is a gap at the top corner along the view direction associated with the special viewpoint. This trick, used by Gregory [4] and Fukuda [9], is called the discontinuity trick.

The discontinuity trick can also be used to



(a) (b)

Figure 3: Realization using discontinuity trick.

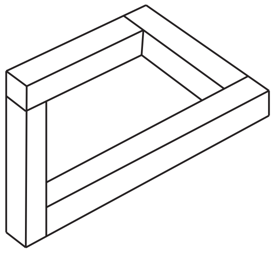


Figure 4: Draper's rectangle.

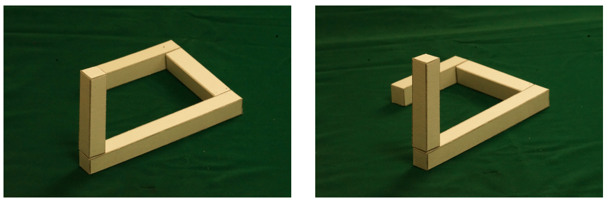
realize the image in Figure 1(a). This figure consists of three faces. We can construct these three faces using planar boards independently and place them in the space in such a way that their boundary lines align when seen from a special viewpoint. However, this realization is less of creativity because any figure composed of closed faces can be trivially constructed using this strategy.

For the realizations in Figures 2 and 3, it is difficult to place the object for exhibition. The curved surface creates nonuniform shading, as shown in Figure 2(a), based on which the trick can be easily guessed even if the figure is viewed from the correct viewpoint. The discontinuous gaps can be seen when the viewpoint is even slightly changed. Therefore, another method is desirable for realizing 3D objects.

3 Draper's Rectangle

One reason why the discontinuity trick easily breaks down is that the hidden part in the image is nearer to the viewer than the obstructing part in the 3D object. Note that in the impossible figure in Figure 1(b), the top of the vertical rod is hidden by the upper-right rod, but in the actual object shown in Figure 3, the vertical rod is nearer to the viewer than the upper-right rod. Because of this, the discontinuity can be easily seen. The trick would be less visible if the hidden part in the image were farther than the obstructing part in the actual 3D object. This can be realized for the impossible figure proposed by Draper [17], which is shown in Figure 4.

This figure is a generalization of the Penrose triangle to a rectangle; four rectangular rods are connected to form a closed ring, but they are connected in such a manner that the structure seems to be physically impossible. For this figure, we can construct a 3D object using the discontinuity trick in such a way that the hidden part in the image is farther than the hiding part in the actual 3D object. The realization with this property is shown in Figure 5. The discontinuity is placed at the upper-right corner, where the obstructing rod is nearer than the hidden rod when we connect the rods at other corners in a natural way. As a result, the trick is slightly less recognizable. Indeed, when the viewpoint is slightly changed, the relation between the obstructing part and the hidden part is not disturbed and thus the sense of impossibility remains. However, this is a lucky exception for this impossible figure.

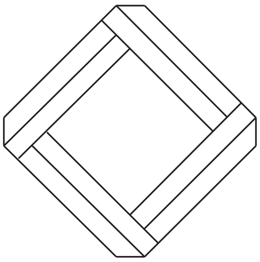


(a) (b)

Figure 5: Realization of Draper's rectangle using discontinuity trick.

4 Penrose Polygons and Their Realization Using Non-Rectangularity Trick

A natural generalization of the Penrose triangle is to generalize the regular triangle to a regular polygon. Two examples are shown in Figure 6, where (a) is a rectangular generalization and (b) is a pentagonal generalization. In both figures, the configuration at the corners is the same as that of the Penrose triangle except for the angle between the rods. This generalization is possible for a regular n-gon for any $n \geq 3$. The generalized figure to a regular n-gon is denoted as a *Penrose n-gon*.



(a) (b)

Figure 6: Penrose rectangle and pentagon.

We can construct a 3D object from the Penrose rectangle using the discontinuity trick in the same manner as that for the Penrose triangle. However, the resulting object has a discontinuous gap where the obstructing part is farther than the hidden part from the viewer. Therefore, the trick is very sensitive to viewpoint. We want a more stable realization that is more robust to changes in viewpoint.

To this end, we employ a trick in which faces that look planar are made using actually planar faces and parts that look connected are actually connected [12, 18]. This trick does not use curved surfaces or discontinuities. Instead, it uses arbitrary angles between adjacent faces even if they look orthogonal. This trick is called the non-rectangularity trick. It is known that the human brain prefers right angles when interpreting images as 3D objects [19]. If we use non-rectangular angles to connect faces, the brain will try to interpret them as right angles. This gives the impression that some figures are physically impossible even though they are realizable as 3D structures.

The original method for constructing 3D objects using the non-rectangularity trick employs a system of equations [12]. For Penrose n-gons, we can construct 3D objects in a more intuitive manner, as described below. As shown in Figure 7, we assign ordinal numbers to the rods, from 1 to n counterclockwise. We name the two visible faces as f_{i1} and f_{i2} and the five vertices as $v_{i1}, \cdots, v_{i5}$ for $i = 1, \cdots n$. We fix an xyz Cartesian coordinate system in such a way that the image is on the xy plane, the origin is at the center of the figure of the Penrose regular polygon, the positive direction of the z axis is oriented toward the front side of the paper, the viewpoint is at $\mathrm{P} = (0, 0, d)$ for $d > 0$.

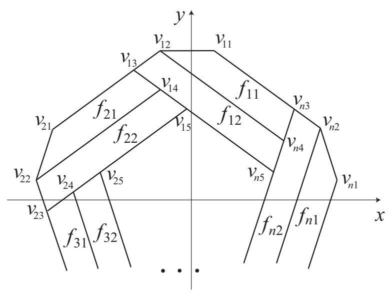


Figure 7: Face and vertex numbers assigned to Penrose n-gon.

Our goal is to construct a 3D object whose central projection onto the xy plane with respect to the center of the projection at P matches the target figure. For this purpose, we assign the z coordinates to the vertices as follows. First, for all vertices v_{i3} $(i = 1, \cdots, n)$, we fix their z coordinates as

$$z_0 = 0 \tag{1}$$

and for all vertices v_{i4} $(i = 1, \cdots, n)$, we fix their z coordinates as

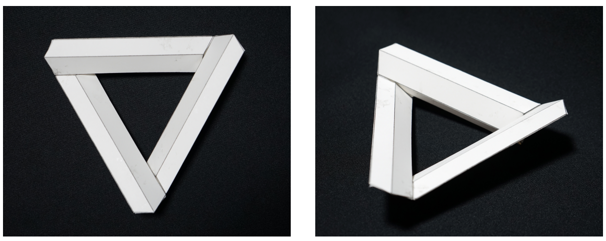
$$z_1 = a, \tag{2}$$

where a is a negative constant.

Then, the visible part of the object is determined uniquely by the following process. In what follows, we read $i+1$ as 1 for $i = n$, and $i-1$ as n for $i = 1$. Each vertex should be on the half line starting at the viewpoint P and passing through the associated point on the image. Hence, if we give the z value of a vertex, it is fixed in the space uniquely. Moreover, note that the faces f_{i2} and $f_{i+1,1}$ are connected and hence the vertices v_{i3} and v_{i4} are on both of the faces.

First, the three vertices v_{i3}, v_{i4}, and $v_{i-1,4}$ are fixed in space based on equations (1) and (2). The face f_{i2}, which contains these three vertices, is thus also fixed in space. Then, the other two vertices v_{i2} and $v_{i-1,5}$ are fixed as the intersections between this face and the half lines starting at P and passing through these vertices. Next, because the vertices v_{i2}, $v_{i-1,3}$, and $v_{i-1,4}$ are already fixed in space, the face f_{i1} and thus vertex v_{i1} can be determined in space. The visible structure is thus placed in the space. Note that vertex v_{i5} (which is on face f_{i2}) is farther than face $f_{i-1,2}$ from the viewer because of equations (1) and (2). Also note that the resulting 3D structure is symmetric with respect to a rotation by $2\pi/n$ around the z axis.

Figure 8 shows a paper object that realizes the Penrose triangle using the non-rectangularity trick, where (a) shows the object from a special viewpoint and (b) shows the same object from a general viewpoint. As shown in the figure, all the faces are planar and there is no discontinuity between parts that look connected in the original impossible figure. Moreover, the structure is rotationally symmetric, which is expected from the original figure. This object is thus less sensitive to changes in viewpoint, making it more suitable for exhibitions.



(a) (b)

Figure 8: Realization using non-rectangularity trick.

Figure 9 shows a diagram of the unfolded surface used for a rod of the Penrose triangle. We can construct the 3D object shown in Figure 8 using three copies of this diagram. The gray areas are used for gluing. This diagram is cut, mountain-folded along the solid lines, and glued to form a rod. The broken lines in the diagram represent the area on which the neighboring rod is to be glued. Face $f_{i+1,3}$ should be glued to this area. The three rods are glued in the same manner to produce the 3D realization shown in Figure 8. The designed distance from the center of the object to the viewpoint is equal to the length of the edge between the faces f_{i1} and f_{i2} multiplied by 3.52.

Three more examples of reconstruction using

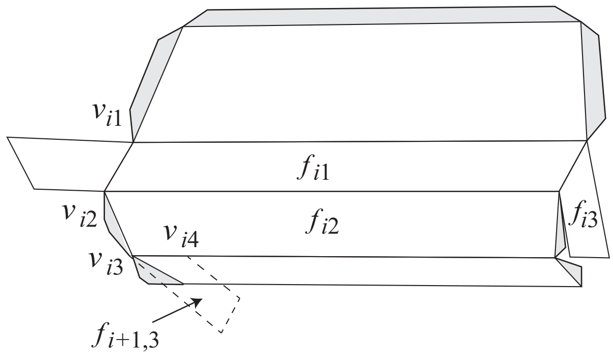
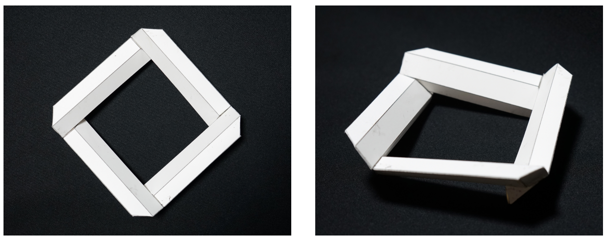


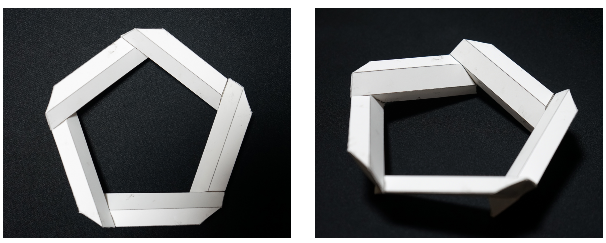
Figure 9: Unfolded surface diagram of rod for Penrose triangle.

the non-rectangularity trick are presented below. Figure 10 shows a 3D realization of the Penrose rectangle in Figure 6(a), Figure 11 shows a 3D realization of the Penrose pentagon in Figure 6(b), and Figure 12 shows a 3D realization of the Penrose hexagon. We can construct 3D objects for all Penrose n-gons in a similar way.



(a) (b)

Figure 10: Realization of Penrose rectangle.

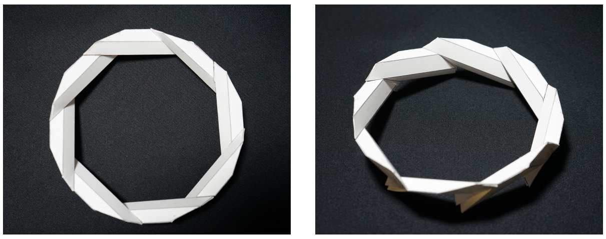


(a) (b)

Figure 11: Realization of Penrose pentagon.

5 Gradient Space Representation

A graphical characterization is known for the realizability of a 3D polyhedral structure from 2D



(a) (b)

Figure 12: Realization of Penrose octagon.

images [14, 15, 16]. In this section, we discuss our method from this graphical characterization point of view. In the following, we assume that a 2D image is an orthographic projection of a 3D object, which is a special case where the viewpoint is at infinity. However, the discussion is valid because the realizability for the central projection is equivalent to the relizability for the orthographic projection, although the relized object shapes are different [12].

Let f_1 and f_2 be two planes fixed in the xyz space. Suppose that they are represented by

$$z = a_i x + b_i y + c_i \tag{3}$$

for $i = 1, 2$. Vector (a_i, b_i) is called the *gradient* of the plane f_i. Any planes with the same gradient are parallel to each other. The gradient (a_i, b_i) can be regarded as a point in the two-dimensional plane. This plane is called the *gradient space*.

Let L be the line obtained when we project the intersection of the two planes onto the xy plane orthographically. The line L is obtained if we eliminate z from equation (3) for $i = 1, 2$. Thus, we get

$$(a_1 - a_2)x + (b_1 - b_2)y = c_2 - c_1. \tag{4}$$

Equation (4) implies that when we overlay the xy plane and the gradient space, line L will be perpendicular to the line connecting the two gradients (a_1, b_1) and (a_2, b_2); see Figure 13. When we consider half planes bounded by the line of intersection instead of the whole planes, we can also distinguish between a convex connection and a concave connection with respect to the viewer. If the connection is convex, one moving in the direction from (a_1, b_1) to (a_2, b_2) crosses line L from area f_1 to area f_2. If the connection is concave, one moving in the direction from (a_1, b_1) to (a_2, b_2) crosses line L from area f_2 to area f_1.

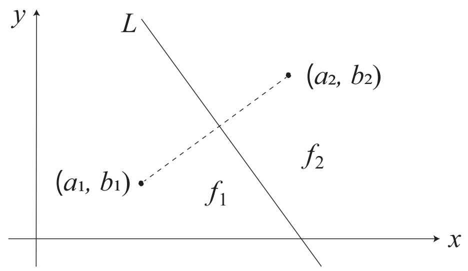


Figure 13: Gradients and intersection of planes.

Therefore, if a line drawing that represents a structure composed of planar faces is realizable as a 3D object, the gradient points associated with the faces can be located in the xy plane in such a way that the line connecting two gradient points is perpendicular to the line of the intersection of the corresponding two faces, and they are located either in the same or opposite order according to whether the edge is convex or concave. This is a graphical way to check the realizability of a 3D object. However, being able to draw the diagram of gradient points is a necessary but not sufficient condition. Hence, it is used to find actually impossible figures, but if the graphical condition is fulfilled, we need to check whether the positions of the edges are consistent using some other methods.

The diagram in the gradient space is nevertheless useful for understanding the associated 3D structures intuitively. Figure 14 shows the gradient diagram for the Penrose pentagon, where g_{i1} and g_{i2} represent the gradient points corresponding to the faces f_{i1} and f_{i2}, respectively, for $i = 1, \cdots, 5$. Note that the edge between f_{i1} and f_{i2} is perpendicular to the line connecting g_{i1} and g_{i2}, and the edge between f_{i2} and $f_{i+1,1}$ is perpendicular to the line connecting $g_{i,2}$ and $g_{i+1,1}$. We can see that the direction from f_{i1} to f_{i2} is the same as the direction from g_{i1} to g_{i2}, which corresponds to the fact that these two faces have a convex connection. In contrast, the direction f_{i2} to $f_{i+1,1}$ is opposite to the direction from g_{i2} to $g_{i+1,1}$, which corresponds to the fact that these two faces have a concave connection. If we fix the center of the coordinate system at the center of this diagram, then the associated 3D realization will be rotationally symmetric with respect to a rotation by $2\pi/5$ ($2\pi/n$ for the general Penrose n-gon).

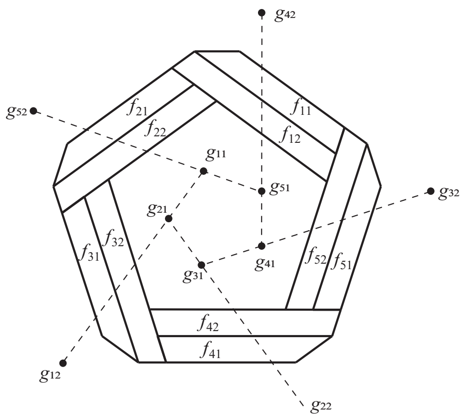


Figure 14: Gradient diagram for Penrose pentagon.

Similar rotationally symmetric diagrams can be drawn for any Penrose polygon, from which the orientations of the faces can be intuitively understood.

6 Concluding Remarks

We proposed a method for realizing 3D polyhedral structures from a series of impossible figures called Penrose polygons. This methods does not employ curved surfaces or discontinuities. Instead, it uses the non-rectangularity trick, in which arbitrary angles are used between faces even though they appear to meet at right angles. The resulting 3D objects are less sensitive than existing methods to viewpoint in the sense that the trick does not break down when the viewpoint slightly changes. Moreover, we can make the 3D structure rotationally symmetric, preserving the elegance of the original impossible figure.

The Penrose triangle is one of the most beautiful impossible figures because of its simplicity, symmetry, and strong sense of impossibility. The figure and its 3D realizations have been used in various fields, including painting, trick sculpture, visual psychology, and solid modeling. The proposed realization method could accelerate the use of 3D realizations because it can generate more stable visual effects. New applications of this method will be considered in future research.

Acknowledgments

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