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Modeling of Ambiguous Tiling for Mold Casting

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Abstract

This paper presents a method for designing 3D ambiguous objects that can be produced by mold casting. The ambiguous objects are 3D solids that give two quite different appearances when seen from two special viewpoints. They can be constructed by 3D printers, but cannot necessarily be constructed by die casting and hence are not suitable for mass production. In this paper we concentrate on one class of ambiguous objects called ambiguous tilings, and propose a method for designing 3D objects that can be tapered monotonically and consequently can be drawn from the mold. This method enables us to apply the ambiguous tiling to real-life uses such as artistic fences and lattice windows.

1 Introduction

The optical illusion seems one of promising sources of visual effects for developing art, and actually a new movement called Op Art started in the twentieth century, where B. Riley and V. Vasarly are pioneers [1] [2]. They utilized anomalous motion illusion and brightness illusion in their artworks to create new visual effects efficiently.

Another example is "tiling," where figure units called tiles cover the whole plane without gaps or overlaps. M. C. Escher created various complicated tiles representing humans and animals to form beautiful tiling patterns [3]. These patterns create visual illusion called figure-ground reversal, where focused figures and surrounding backgrounds exchange their roles alternatingly. His activity can be regarded as a revolution in tiling art because tiling were traditionally made using abstract geometric tiles such as parallelograms and regular polygons.

Still another direction of applications of optical illusion is "ambiguous objects," where the same objects produce multiple interpretations. Early trials include Fukuda's multiple-silhouette sculptures such as "Encore" (1976), which gives a silhouette of a pianist and that of a violinist. Ohgami and Sugihara [4] tried to extend the multiple-silhouette objects from two silhouettes to three by a heuristic method, and Mitra and Pauly [5] formulated a optimization method for finding three-silhouette objects. Hsiao et al. [6] considered a similar optimization method for wireframe structures that create three desired line drawings as silhouettes.

A different class of ambiguous objects was proposed by Sugihara [7]. An object in this class looks like a cylinder, but the section of the cylinder changes its appearance when it is seen from opposite side. This class has been extended to several directions, including topology-disturbing objects [8], anomalous-symmetry objects [9] and reflexively-fused objects [10].

Sugihara [11] combined the tiling art and the ambiguous cylinders to create ambiguous tiling, where the same object creates two tiling patterns when it is seen from two special viewpoints. He created many examples of ambiguous tiling using 3D printers. This class of objects has large potential application areas such as artistic fences, lattice windows and donation boxes. However, we cannot apply the objects directly, because they are not suitable for mass production by mold casting.

In this paper, we propose a new method for designing 3D shapes of ambiguous tiling so that the resulting shapes are drawable from the mold and hence can be constructed by mold casting. We review the ambiguous tiling (section 2), present the difficulty to be overcome (section 3), propose a new method (section 4), and give concluding remarks (section 5).

2 Review of Ambiguous Cylinders and Tiling

We briefly review the concepts of the ambiguous cylinders and the ambiguous tiling, placing emphasis on the computational method for the 3D cylindrical object. These are the basis on which we propose our new solid modeling method.

2.1 Ambiguous Cylinders

An ambiguous cylinder is a cylindrical object whose section creates two desired appearances when it is seen from two special directions. As shown in Figure 1, we fix an (x, y, z) Cartesian coordinate system, and give two x-monotone curves $a(s)$ and $b(x)$ on the xy plane defined for $x \in [x_0, x_1]$. Let v_1 and v_2 be two view directions which are parallel to the yz plane and form angles α and β, respectively, with the line parallel to the y axis.

We want to construct a space curve, say $c(x)$, that coincides with the plane curve $a(x)$ when seen in the view direction v_1 and coincides with $b(x)$ when seen in the view direction v_2. For each x, we construct the line parallel to v_1 passing through $a(x)$ and the line parallel to v_2 passing through $b(x)$, and find the point of intersection of the two lines. Let this point be $c(x)$. Thus, we get the space curve $c(x)$. We call $c(x)$ the *ambiguous curve* generated by $a(x)$ and $b(x)$. Finally we translate the curve $c(x)$ in the direction parallel to the z axis by distance h, and adopt the swept surface S. This is the cylindrical surface we want. Indeed when we see S in the direction v_1, we usually perceive the top curve $c(s)$ as the plane curve parallel to the xy plane because S looks a cylinder with the uniform height h obtained by cutting the cylinder with the plane perpendicular to the cylinder axis. This is because the human brains prefer rectangles strongly [12]. We can construct a closed cylinder if we apply this method twice, once for the upper boundary curve and once more for the lower boundary curve.

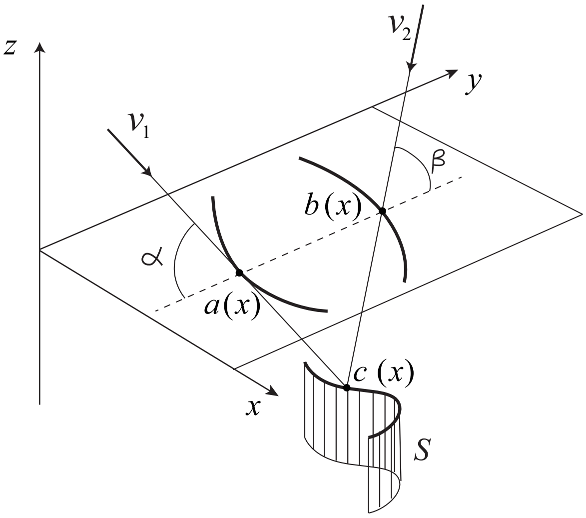


Figure 1: Computation of the ambiguous space curve and the associated cylinder.

Figure 2 shows an example of an ambiguous cylinder. A vertical mirror is placed behind the cylinder. The object itself appear to have a section of a full moon shape, while in the mirror the section appear to be a star.

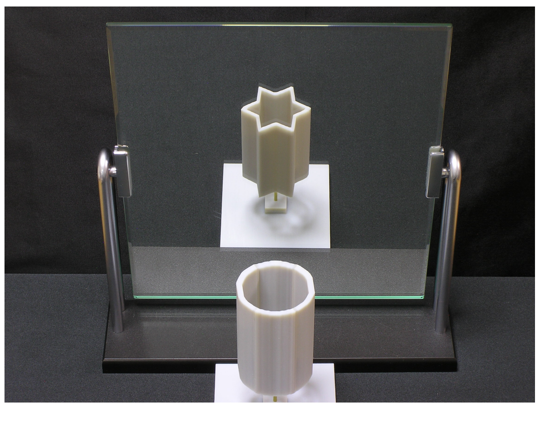


Figure 2: Ambiguous cylinder "Full Moon and Star."

From the construction method shown in Figure 1, we understand that the ambiguous cylinder can be constructed both if (1) there is a one-to-one correspondence between the two given curves in such a way that the corresponding pair of points have the same x coordinate, and if (2) corresponding pairs with the same x coordinate have the same orders in the y direction [7].

2.2 Ambiguous Tiling

A tiling is a planar pattern composed by tiles without gaps or overlaps. A tiling is called *monohedral* if all the tiles are of the same shape. A monohedral tiling is called *isohedral* if the relative placement rules among a tile and the surrounding tiles are the same for all the tiles. It is know that the placement rules of the isohedral tilings are classified into 17 groups, and the graph structures of the isohedral tilings are classified into 93 isomorphic patterns IH1, IH2, ..., IH93 [13]. Figure 3 shows an example of an isohedral tiling named IH55, where tiles are of the shape of a ginkgo leaf, and they are placed by rotations of 90 degrees and translations in two directions.

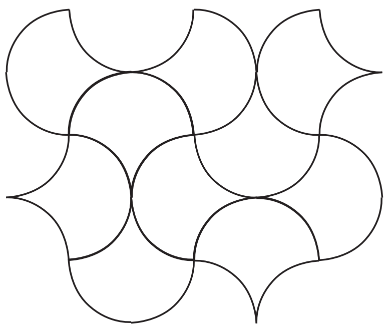


Figure 3: Isohedral tiling IH55 composed of ginkgo-leaf tiles.

In order to make an ambiguous tiling, we have to find a pair of tiles A and B such that (1) both A and B create isohedral tilings and (2) ambiguous cylinders can be created for all pair of corresponding postures of tiles that appear in the tiling. We refer to Sugihara [11] for a general theory; here let us show an intuitive way of constructing the ambiguous tiling by an example.

As shown in Figure 4, consider a pair of tiles fixed to the xy plane. We can construct the ambiguous cylinder from them because the leftmost point of the two tiles have the same x coordinate, the rightmost points have the same x coordinate, and the upper and lower boundaries of the two tiles are all x-monotone. The resulting ambiguous cylinder is shown by CG images in Figure 5, where (a) and (b) show the appearances seen in the directions v_1 and v_2, and (c) shows an appearance from a general viewpoint. Note that the height of the cylinder is short; this is because we consider applications to grid structures such as artistic fences and lattice windows.

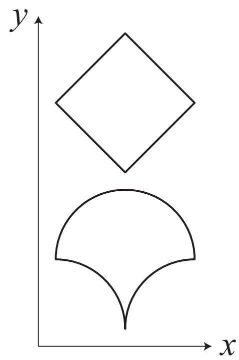


Figure 4: Pair of tiles for the ambiguous cylinder construction.

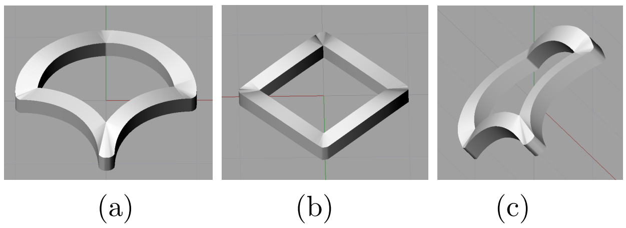
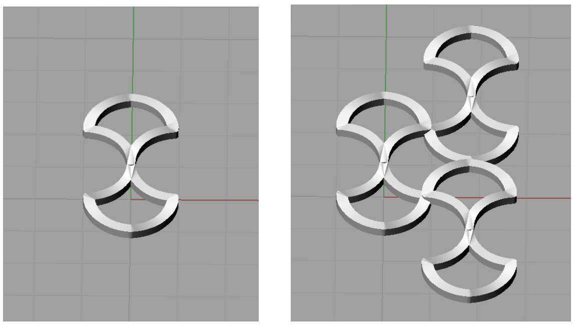


Figure 5: Ambiguous cylinder created from the tiles in Figure 4.

We can construct an ambiguous tiling in the following way. First, we rotate a copy of the cylinder around the x axis by 180 degrees. The rotation implies that we see the cylinder from the back side. Hence the resulting cylinder gives the upside-down version of the ginkgo leaf shape. Next, we connect it to the original cylinder, as shown in Figure 6(a). Then, we make two copies of the object and translate in the upper right and the lower right to construct the structure as shown in Figure 6(b).

Note that the 90-degree rotated version of the ginkgo leaf appears at the center automatically although we do not provide the tile in this pos-



(a) (b)

Figure 6: Construction of the ambiguous tiling.

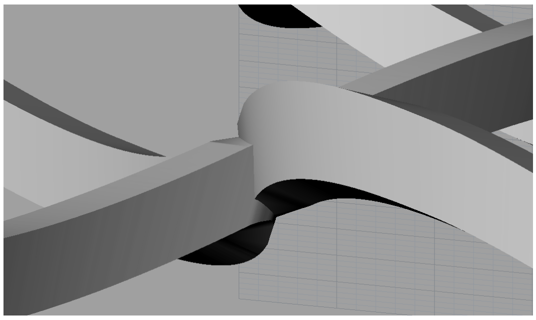
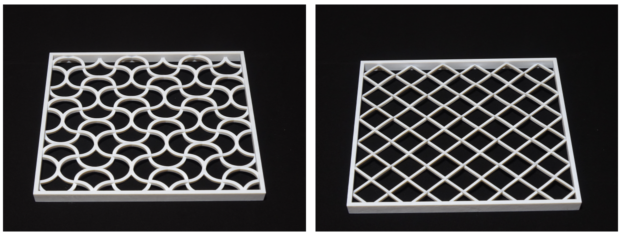


Figure 8: Connection of elements tapered in the same direction.

ture. Repeating this process, we get the tiling composed of ginkgo leaves shown in Figure 3. The edges of the ginkgo leaves appear to be straight line segments forming a square when we see the structure in the second view direction, and hence we get the square grid tiling. Figure 7 shows the associated two appearances of the resulting tiling made by a 3D printer.



(a) (b)

Figure 7: Resulting ambiguous cylinder made by a 3D printer

3 Castability: Problem to Solve

We can construct the physical models of the ambiguous tiling directly using a 3D printer, but it requires high cost and is not suitable for applications.

A typical method for mass production is mold casting. However, the object shapes are restricted to those that can be drawn from the mold. A single ambiguous cylinder can be cast because it can be tapered along the axis of the cylinder and consequently can be drawn from the mold. However, general objects such as ambiguous tilings cannot necessarily be cast, because two or more components are combined and overhang structure may appear even if the axes of the cylinders align.

Figure 8 is a magnified image of the ambiguous tiling in Figure 7 at which two downward tapered cylinders are joined. We can see that the right arm overhangs the left arm and consequently prevents us from drawing the structure downward when it is covered by the mold.

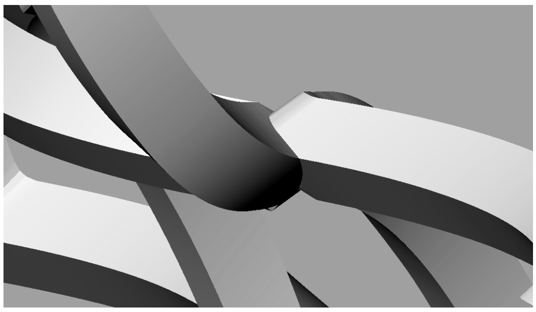


Figure 9: Connection of elements tapered in the opposite direction.

One idea to overcome this difficulty might be to taper every other cylinder in the opposite directions alternately; one downward and its neighbors upward. In that case we should join the elements so that the start lines of the taper meet as shown in Figure 9. Then, the mold can be drawn half and half in both directions, thus the structure can be cast. However, it is in general impossible because one element should be joined to the surrounding elements at many points, e.g., four points in the case of the tiling in Figure 7; if we adjust at one joint, then the conditions are disturbed at other joints. Thus, it is a nontrivial problem to convert the ambiguous tilings so that they can be cast.

4 One-Piece Construction Method

It is difficult to make the structures to fit the mold casting by combining ambiguous cylinder pieces; we need to change the strategy completely. So we give up joining pieces; instead we try to make the whole tiling structure in one piece.

We fix the number n of tiles in the tiling and give a pair of diagrams representing the whole structure of the desired appearances as shown in Figure 10, where $n = 61$. Each of the two diagrams has one outmost boundary and n hole boundaries. We construct the ambiguous tiling by the next method. Details at each step will be discussed later.

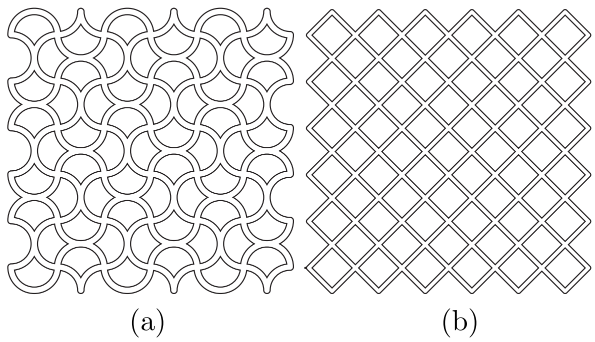


Figure 10: Pair of curve sets for one-piece construction.

One-Piece Construction Method

1. For the pair of outer boundaries, establish a one-to-one correspondence between the points with the same x coordinates, and construct the ambiguous space curve c_0.

2. For n pairs of the corresponding hole boundaries, establish a one-to-one correspondence and construct the ambiguous space curves c_1, c_2, ..., c_n.

3. Create continuous surface M spanning the narrow area surrounded by c_0 and outside of c_1, ..., c_n.

4. Translate M in the z direction by distance h, and create the solid swept by M, and output it.

For Step 3, we utilize the constrained Delaunay triangulation [14] [15]. We represent the ambiguous curves by point sequences and triangulate the area using those points as vertices and the edges connecting consecutive points as the constraints.

In order to get a tapered version of the object, we shrink M during the translation in Step 4. The amount of the taper is specified by the draft angle θ, which is typically 1 or 2 degrees for the sand molding. We define t such that $\tan\theta = t/h$. We shrink the boundaries of M (the outer boundary and the hole boundaries) by the distance t linearly during the translation by the distance h. Thus we get the tapered object with the desired draft angle θ. By this way we can create an object that can be cast.

Hence, the remaining problem is how to establish the one-to-one correspondences in Steps 1 and 2. This is not trivial. For the diagrams of the ginkgo tiling and the square tiling in Figure 10, pairs of holes shown as A and B in Figure 11 do not span the same x interval. Hence we cannot establish the correspondence.

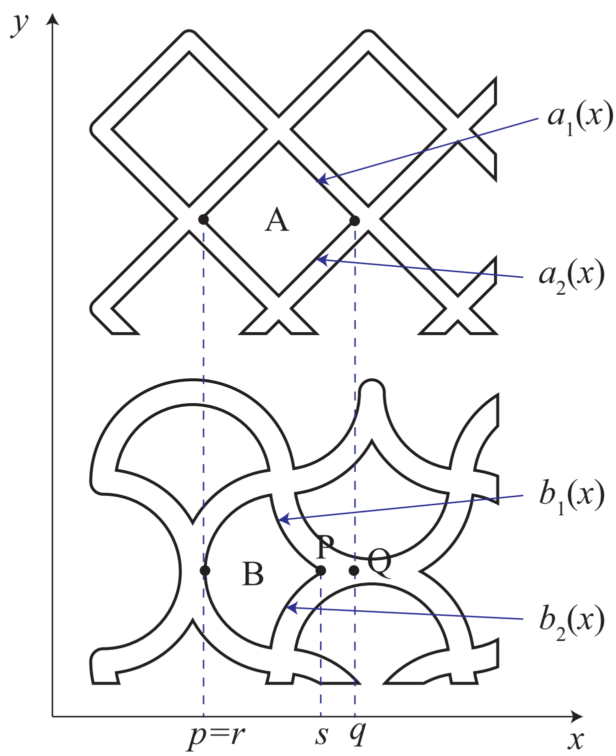


Figure 11: Lost of the one-to-one correspondence.

Suppose that the boundary of the hole A consists of the upper curve $y = a_1(x)$ and the lower curve $y = a_2(x)$ for $x \in [p, q]$, and the boundary of the hole B consists of the upper curve $y = b_1(x)$ and the lower curve $y = b_2(x)$ for $x \in [r, s]$. We assume that $p = r$ and $s < q$, that is, the leftmost point coincide while the rightmost points do not. Hence, the one-to-one correspondence does not exist in $x \in [s, q]$. In order to make the correspondence, we modify the boundary of the hole B.

Let the rightmost point of B be P. We choose a point Q=(q, t) avoiding holes. A simplest way is to choose $t = b_1(s)$, as shown in Figure 11. If this point is in a hole, we move it upward or downward accordingly keeping the x coordinate unchanged. As shown in Figure 12, we choose two curves $y = b_3(x)$ and $y = b_4(x)$ connecting P and Q such that

$$b_4(x) \geq b_3(x), x \in [s, q] \tag{1}$$

and

$$b_4(x) - b_3(x) \leq a_1(x) - a_2(x), x \in [s, q] \tag{2}$$

We extend the curves $y = b_1(x)$ to $y = b'_1(x)$ by connecting it to $y = b_3(x)$, and similarly extend the curve $y = b_2(x)$ to $y = b'_2(x)$ by connecting it to $y = b_4(x)$. As the result, te boundary curves of the hole B span the same range as $y = a_1(x)$ and $y = a_2(x)$, and hence we can make the one-to-one correspondence.

We modify the boundaries by applying the same augmentation at every rightmost extreme point and leftmost extreme point. Then, we get the next theorem.

Theorem 1.

The ambiguous space curve generated by $y = a_1(x)$ and $b'_1(x)$, and that generated by $y = a_2(x)$ and $y = b'_2(x)$ give the desired appearances when seen in the directions v_1 and v_2, Moreover, the resulting 3D object can be cast.

Rough Sketch of the Proof. Because the inequality (1) holds, the associated ambiguous space curve coincides with the desired appearances of the hole boundary. Indeed, the extended part of the curve is cancelled because of the overlap of the surfaces. Because the inequality (2) holds, the projection of the ambiguous space curve onto the xy plane does not cross. Hence the object can be cast. Q.E.D.

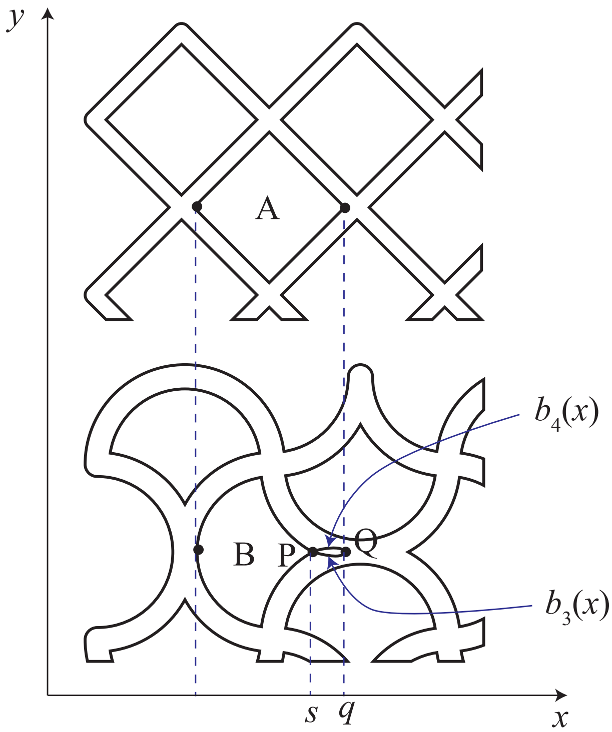


Figure 12: Strategy for establishing the one-to-one correspondence.

For the pair of desired appearances of the tiling in Figure 10, we extend the boundary curves by adding horizontal line segments, i.e., we set $b_3(x) = b_4(x) = b_1(s)$ for $x \in [s, q]$. Figure 13 shows the resulting object seen in the view direction v_1. We can see the tiling composed of ginkgo leaves. Figure 14(a) shows part of this object seen from slightly different view direction. We can see that the upper surface is continuous and has no overhangs, and the shape of the ginkgo leaf is realized. Thus we get the ambiguous tiling that can be cast. Figure 14(b) shows the same part viewed in the z-direction, from which we can see that the boundary-extended part is not so narrow or sharp as we feel when we see (a). So the resulting object has enough strength in this case. This object is now being sand mold casted, and will be used as the top cover of a donation box at a shrine in Takayama City, Japan.

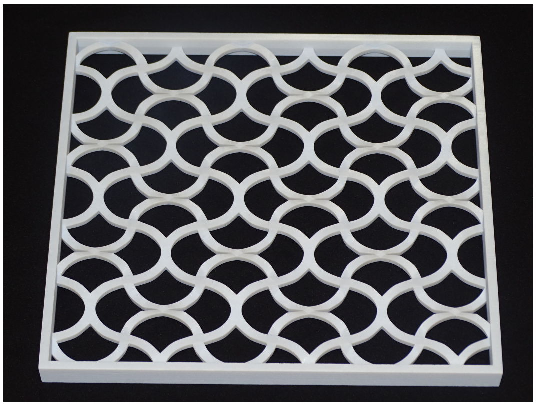
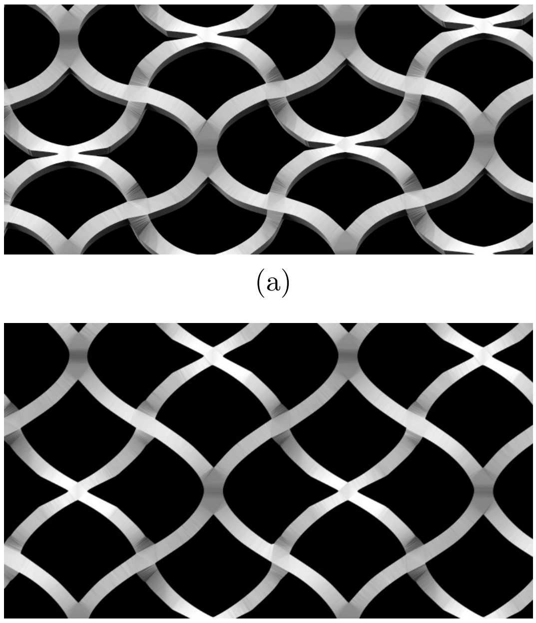


Figure 13: Ambiguous tiling created by the one-piece construction method.



(a)

(b)

Figure 14: Magnified image of the object in Figure 13: (a) part seen from a slightly different viewpoint; (b) that seen in the direction parallel to the z axis.

5 Concluding Remarks

The ambiguous tiling is a new style of art, which is the combination of tiling and ambiguous cylinders. However, the original ambiguous tilings cannot necessarily be cast and consequently not suitable for mass production. In order to overcome this difficulty, we have proposed a one-piece construction method for designing the ambiguous tilings that can be cast.

In this method we compute the whole structure of the tiling at once instead of creating pieces and combining them. This is not straightforward because the one-to-one correspondence between the two goal shapes is lost when we move from individual pieces to the whole structure. To circumvent the difficulty, we augment the boundary in such a way that the augmented part will be hidden by other part of the object and hence the visual effect is not disturbed. Furthermore, we have presented the condition for the resulting object to be created by mold casting.

The ambiguous tilings are new 3D art that have two kinds of visual effects based on optical illusion. One is the tiling, in which the space is covered by elementary pieces so that the figure-ground illusion is evoked in such a way that the focused shape and the surrounding background exchange their roles alternatingly. The other is ambiguity in interpretation in the sense that two quite different tiling patterns are perceived if we change the view direction. Because of these rich visual effects, they have large potential of real applications such as protective fences, stained glasses, lattice walls, and decorative windows. The method proposed in this paper pushes one step to those applications.

Acknowledgments

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